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LETTER TO THE EDITOR

On a Bethe-ansatz approach to the derivative non-linear Schrödinger equation

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Abstract. We continue our analysis of extended derivative non-linear Schrödinger equation with the help of the Bethe-ansatz technique. In our previous analysis by the QISM approach it was not possible to consider the reduction of the extended system to the derivative NLSE, as the theory then becomes non-ultralocal. Therefore, here we have applied the approach of the Bethe ansatz to both the extended DNLSE and the usual DNLSE, and show that it is possible to construct multiparticle quantum states in both the cases.

In a recent paper [1] we have formulated the quantum inverse spectral method for the extended derivative non-linear Schrödinger equation, by following the methodology of Faddeev [2]. At this stage we should explain what actually is meant by the word 'extended'. The original derivative non-linear Schrödinger equation (DNLSE) is known to be non-ultralocal and non-canonical and Faddeev's approach of QISM is not applicable to such a case. On the other hand, the heirarchy of equations derived by Gerdjikov and Ivanov [3] contains a coupled set of NLSE which under reduction goes over to DNLSE. But this 'extended' coupled set is canonical and ultralocal, and so it was possible to apply the QISM directly to this set but not to the original DNLSE. Furthermore, even after quantisation it was not possible to observe the fate of the reduced DNLSE within the set-up. Therefore we have addressed ourselves to the question of analysing further quantum mechanical aspects of both the extended and reduced systems, and incidentally the most elegant and alternative approach to QISM is that of the Bethe ansatz [4]. Therefore in the following we discuss these two equations on the basis of the Bethe ansatz.

The extended DNLSE is generated by the Hamiltonian

$$H=\tfrac{1}{2}\int_{-\infty}^{\infty} dx[(\varepsilon_0 q_{1xx}q_0^* + \varepsilon_1 q_{0xx}q_1^*)+\tfrac{3}{5}i\varepsilon_1\varepsilon_0 q_{1x}q_1q_1^*q_0^* - \tfrac{3}{5}i\varepsilon_1^2 q_0q_1q_{1x}^*q_1^* + V_0V_1] \tag{1}$$

where

$$\begin{aligned} V_0 &= \varepsilon_1 q_0 q_1^* + \varepsilon_0 q_1 q_0^* \\ V_1 &= \tfrac{1}{2}\varepsilon_1^2|q_1^4| - 2\varepsilon_0|q_0^2|. \end{aligned} \tag{2}$$

The canonical set of commutation rules are

$$\begin{aligned} [\varepsilon_0 q_0^*(x), q_1(x_1)] &= \delta(x-x_1) \\ [\varepsilon_1 q_0(x), q_1^*(x_1)] &= -\delta(x-x_1). \end{aligned} \tag{3}$$

In the QISM approach [1] we started with the definition of the vacuum, given as

$$q_1^*|0\rangle = q_0^*|0\rangle = 0. \tag{4}$$

Here we proceed to construct the Bethe-like states by assuming condition (4) to be valid.

For a one-particle state let us set

$$|1, 1\rangle = \int \mathrm{d}x_0\, f_1(x) q_1(x)|0\rangle + \int \mathrm{d}x\, f_2(x) q_0(x)|0\rangle \tag{5}$$

and demand $H|1, 1\rangle = E|1, 1\rangle$ which immediately leads to

$$\tfrac{1}{2}\partial^2 f_1/\partial x^2 = E f_1(x) \qquad \tfrac{1}{2}\partial^2 f_2/\partial x^2 = E f_2. \tag{6}$$

Now we consider two-particle states, but before actually writing them down let us clarify the notation we use in designating these states. In general, any such state vector will be a linear combination of states containing m of one species and n of the other species; therefore we specify these numbers in the argument of these states. It follows that the two-particle state can be written as

$$\begin{aligned}|2, 11, 2\rangle = &\iint \mathrm{d}x_1\, \mathrm{d}x_2\, g_1(x_1, x_2) q_1(x_1) q_1(x_2)|0\rangle \\ &+ \iint \mathrm{d}x_1\, \mathrm{d}x_2\, g_2(x_1, x_2) q_1(x_1) q_0(x_2)|0\rangle \\ &+ \iint \mathrm{d}x_1\, \mathrm{d}x_2\, g_3(x_1, x_2) q_0(x_1) q_0(x_2)|0\rangle.\end{aligned} \tag{7}$$

Thus the condition $H|2, 11, 2\rangle = E|2, 11, 2\rangle$ leads to

$$\frac{1}{2}\left(\frac{\partial^2 g_2}{\partial x_1^2} + \frac{\partial^2 g_2}{\partial x_2^2}\right) - \frac{3\mathrm{i}}{10}\left(\frac{\partial g_3}{\partial x_1} + \frac{\partial g_3}{\partial x_2}\right)\delta(x_1 - x_2) - 4 g_1(x_1, x_2)\delta(x_1 - x_2) = E g_2(x_1 x_2) \tag{8a}$$

$$\frac{1}{2}\left(\frac{\partial^2 g_1}{\partial x_1^2} + \frac{\partial^2 g_1}{\partial x_2^2}\right) + \frac{3\mathrm{i}}{10}\left(\frac{\partial g_2}{\partial x_1} + \frac{\partial g_2}{\partial x_2}\right)\delta(x_1 - x_2) = E g_1(x_1 x_2) \tag{8b}$$

$$\frac{1}{2}\left(\frac{\partial^2 g_3}{\partial x_1^2} + \frac{\partial^2 g_3}{\partial x_2^2}\right) - 2 g_2(x_1 x_2)\delta(x_1 - x_2) = E g_3(x_1 x_2). \tag{8c}$$

The three-particle states are

$$\begin{aligned}|3, 12, 21, 3\rangle = &\iiint \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3\, h_1(x_1 x_2 x_3) q_1(x_1) q_1(x_2) q_1(x_3)|0\rangle \\ &+ \iiint \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3\, h_2(x_1 x_2 x_3) q_0(x_1) q_1(x_2) q_1(x_3)|0\rangle \\ &+ \iiint \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3\, h_3(x_1 x_2 x_3) q_0(x_1) q_0(x_2) q_1(x_3)|0\rangle \\ &+ \iiint \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3\, h_4(x_1 x_2 x_3) q_0(x_1) q_0(x_2) q_0(x_3)|0\rangle.\end{aligned} \tag{9}$$

An analysis as in the previous cases leads to

$$\begin{aligned}&\frac{1}{2}\left(\frac{\partial^2 h_1}{\partial x_1^2} + \frac{\partial^2 h_1}{\partial x_2^2} + \frac{\partial^2 h_1}{\partial x_3^2}\right) + \frac{3\mathrm{i}}{10}\left[\left(\frac{\partial h_2}{\partial x_2} + \frac{\partial h_2}{\partial x_1}\right)\delta(x_2 - x_1) + \left(\frac{\partial h_2}{\partial x_3} + \frac{\partial h_2}{\partial x_1}\right)\delta(x_3 - x_1)\right] \\ &\qquad + h_3 \delta(x_1 - x_3)\delta(x_2 - x_3) = E h_1\end{aligned} \tag{10a}$$

$$\frac{1}{2}\left(\frac{\partial^2 h_2}{\partial x_1^2}+\frac{\partial^2 h_2}{\partial x_2^2}+\frac{\partial^2 h_2}{\partial x_3^2}\right)+\frac{3\mathrm{i}}{5}\left[\left(\frac{\partial h_3}{\partial x_2}+\frac{\partial h_3}{\partial x_3}\right)\delta(x_2-x_3)-\left(\frac{\partial h_3}{\partial x_2}+\frac{\partial h_3}{\partial x_1}\right)\delta(x_1-x_2)\right]$$

$$+3h_4\delta(x_2-x_3)\delta(x_1-x_3)-4h_1[\delta(x_1-x_2)+2\delta(x_2-x_3)]=Eh_2 \tag{10b}$$

$$\frac{1}{2}\left(\frac{\partial^2 h_3}{\partial x_1^2}+\frac{\partial^2 h_3}{\partial x_2^2}+\frac{\partial^2 h_3}{\partial x_3^2}\right)-\frac{3\mathrm{i}}{5}\left[\left(\frac{2\partial h_4}{\partial x_3}+\frac{\partial h_4}{\partial x_1}\right)\delta(x_1-x_3)+\left(\frac{2\partial h_4}{\partial x_3}+\frac{\partial h_4}{\partial x_2}\right)\delta(x_2-x_3)\right]$$

$$-4h_2[\delta(x_1-x_2)+\delta(x_2-x_3)]=Eh_3 \tag{10c}$$

$$\frac{1}{2}\left(\frac{\partial^2 h_4}{\partial x_1^2}+\frac{\partial^2 h_4}{\partial x_3^2}+\frac{\partial^2 h_4}{\partial x_3^2}\right)-2h_3[\delta(x_1-x_3)+\delta(x_2-x_3)]=Eh_4. \tag{10d}$$

Next we look for solutions to the wave equations. In the case of one-particle states, the result is obviously simple plane waves. But in the case of two-particle states we assume solutions of the following form subjecting these to proper boundary conditions and the equations themselves. Let us set

$$g_i(x_1, x_2)=A_i\exp[\mathrm{i}(K_1x_1+K_2x_2]\theta(x_1-x_2)+B_i\exp[\mathrm{i}(K_1x_1+K_2x_2)]\theta(x_1-x_2). \tag{11}$$

Substitution of (11) in (8*a*, *b*, *c*) immediately leads to

$$E=-\tfrac{1}{2}(K_1^2+K_2^2) \tag{12}$$

$$\mathrm{i}\tfrac{1}{2}(K_1-K_2)(A_1-B_1)=\tfrac{3}{5}\mathrm{i}[\mathrm{i}\tfrac{1}{2}(K_1+K_2)(A_3+B_3)]-2(A_2+B_2) \tag{13a}$$

$$\mathrm{i}\tfrac{1}{2}(K_1-K_2)(A_2-B_2)=\tfrac{3}{20}(K_1+K_2)(A_1+B_1) \tag{13b}$$

$$\mathrm{i}\tfrac{1}{2}(K_1-K_2)(A_3-B_3)=(A_1+B_1). \tag{13c}$$

If we now impose the usual normalisation conditions for g_i then we obtain

$$\frac{A_1}{B_1}=\frac{K_1(10+6\mathrm{i})-K_2(10-6\mathrm{i})-40\mathrm{i}}{K_1(10-6\mathrm{i})-K_2(10+6\mathrm{i})+40\mathrm{i}} \tag{14}$$

$$\frac{A_2}{B_2}=\frac{K_1(10-3\mathrm{i})-K_2(10+3\mathrm{i})}{K_1(10+3\mathrm{i})-K_2(10-3\mathrm{i})} \tag{15}$$

$$\frac{A_3}{B_3}=\frac{K_1-K_2-2\mathrm{i}}{K_1-K_2+2\mathrm{i}}. \tag{16}$$

A similar analysis holds for the case of three-particle states, but it is not possible to write out the equations for the explicit solutions of the wavefunctions $h_i(x_1x_2x_3)$ as they are large in number (we obtained 22 equations) and so can be solved only with the help of a computer. Therefore, we shall only study the reduced DNLSE itself rather than the extended version of the equation.

In this case the equations are

$$\mathrm{i}u_t+u_{xx}+\varepsilon_1(|u|^2u)_x=0 \tag{17}$$

where the Hamiltonian is

$$H=\int \mathrm{d}x[-\tfrac{1}{2}(uu_x^*-u_xu^*)+\varepsilon_1|u(x)|^2|u(x)|^2]$$

and the basic commutation is

$$[u(x_1), u^*(x)]=\partial(x_1-x)/\partial x_1]. \tag{18}$$

We now proceed to construct the particle states as before by setting

$$|1\rangle = \int \mathrm{d}x_1\, f_1(x_1)\, U(x_1)|0\rangle$$

$$|2\rangle = \int\int \mathrm{d}x_1\, \mathrm{d}x_2\, f_2(x_1x_2)\, U(x_1)\, U(x_2)|0\rangle \tag{19}$$

$$|3\rangle = \int\int\int \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3\, f_3(x_1x_2x_3)\, U(x_1)\, U(x_2)\, U(x_3)|0\rangle$$

and so on.

For the one-particle state we get the obvious result

$$E = -K^2 \qquad \partial^2 f_1/\partial x^2 = Ef_1.$$

For the two-particle state we get

$$\frac{\partial^2 f_2}{\partial x_1^2} + \frac{\partial^2 f_2}{\partial x_2^2} + 2\varepsilon_1 \frac{\partial f_2}{\partial x_1}\frac{\partial}{\partial x_2}\delta(x_2 - x_1) = Ef_2(x_1x_2). \tag{20}$$

We now define new variables:

$$X_{12} = x_2 - x_1 \qquad Y_{12} = x_2 + x_1$$

so we get two separate equations,

$$\frac{\partial^2 \tilde{f}_1}{\partial X_{12}^2} + 2\varepsilon_1 \frac{\partial^2 \tilde{f}_1}{\partial X_{12}^2}\delta(X_{12}) = E_1 \tilde{f}_1 \tag{21a}$$

$$\frac{\partial^2 f_2}{\partial Y_{12}^2} = E_2 \tilde{f}_2 \qquad f_2 = \tilde{f}_1(X_{12})\tilde{f}_2(Y_{12}) \tag{21b}$$

$$E_1 + E_2 = E = -(K_1^2 + K_2^2).$$

Equation (21a) can be solved by the imposition of the boundary condition deduced in the usual way by the integration of (21a) over a region $x_1 = x_2$. In our case this leads to

$$\left(\frac{\partial f_1}{\partial X_{12}}\right)_+ - \left(\frac{\partial f_1}{\partial X_{12}}\right)_- + 2\varepsilon_1\left(\frac{\partial^2 f_1}{\partial X_{12}^2}\right)_{X_{12}=0} = 0$$

and the total solution is

$$\begin{aligned} f(x_1x_2) = {} & \{a_2 \exp[-\mathrm{i}K_2(x_1 + x_2)] + b_2 \exp[\mathrm{i}K_2(x_1 + x_2)]\} \\ & \times ((1+F)\exp[\mathrm{i}K_1(x_1 - x_2)]\theta(x_2 - x_1) \\ & + \{\exp[\mathrm{i}K_1(x_2 - x_1)] + B\exp[-\mathrm{i}K_1(x_2 - x_1)]\theta(x_1 - x_2)\}) \end{aligned} \tag{22}$$

$$B = F = \frac{-\mathrm{i}\varepsilon_1 K_1}{1 + \mathrm{i}\varepsilon_1 K_1}.$$

In the case of three-particle states our equation is

$$\left(\frac{\partial^2 f_3}{\partial x_1^2} + \frac{\partial^2 f_3}{\partial x_2^2} + \frac{\partial^2 f_3}{\partial x_3^2}\right) + 2\varepsilon_1\left(\frac{\partial f_3(x_1x_2x_3)}{\partial x_1}\frac{\partial}{\partial x_2}\delta(x_2 - x_1)\right)$$
$$+ \text{terms with interchange of } (1, 2, 3) = Ef_3. \tag{23}$$

To solve (23) we divide the regions in the coordinate space, and consider the following cases.

Case (*i*): $x_2 \neq x_3$; $x_1 \neq x_3$. Here we define

$$x_2 - x_1 = X_{12} \qquad x_2 + x_1 = Y_{12} \qquad x_3 = Z_{12}.$$

Then we get

$$\frac{\partial^2 f_3}{\partial X_{12}^2} + \frac{\partial^2 f_3}{\partial Y_{12}^2} + \frac{\partial^2 f_3}{\partial Z_{12}^2} + 2\varepsilon_1 - \frac{\partial f_3}{\partial X_{12}} \frac{\partial}{\partial X_{12}} \delta(X_{12}) = E f_3.$$

So here also

$$\frac{\partial^2 f_3}{\partial X_{12}^2} + 2\varepsilon_1 \frac{\partial^2 f_3}{\partial X_{12}^2} = E_1 f_1$$

$$\frac{\partial^2 f_3}{\partial Y_{12}^2} = E_2 f_3$$

$$\frac{\partial^3 f_3}{\partial Z_{12}^2} = E_3 f_3$$

$$E = E_1 + E_2 + E_3 = -(K_1^2 + K_2^2 + K_3^2).$$

The situation is similar to the two-particle case and we have a solution of the form

$$\begin{aligned} f_3 = [a_3 \exp(-\mathrm{i}K_3 x_3) + b_3 \exp(-\mathrm{i}K_3 x_3)]\{a_2 \exp[-\mathrm{i}K_2(x_1 + x_2)] + b_2 \exp[\mathrm{i}K_2(x_1 + x_2)]\} \\ \times((1 + F(K_1)) \exp[\mathrm{i}K_1(x_2 - x_1)]\theta(x_2 - x_1) + \{\exp[\mathrm{i}K_1(x_2 - x_1)] \\ + B(K_1) \exp[-\mathrm{i}K_1(x_2 - x_1)]\}\theta(x_1 - x_2)). \end{aligned} \tag{24}$$

Case (*ii*): $x_2 \neq x_1$, $x_3 \neq x_1$. Here we define

$$X_{23} = x_3 - x_2 \qquad Y_{23} = x_3 + x_2 \qquad Z_{23} = x_1.$$

Here the reduced equation can be seen to be

$$\begin{aligned} &\frac{\partial^2 f_3'}{\partial X_{23}^2} + 2\varepsilon_1 \frac{\partial^2 f_3'}{\partial X_{23}^2} \delta(X_{23}) = E_1' f_3' \\ &\frac{\partial^2 f_3'}{\partial Y_{23}^2} = E_2' f_3' \\ &\frac{\partial^2 f_3'}{\partial Z_{23}^2} = E_3' f_3' \end{aligned} \tag{25}$$

and the solution can be rewritten as in (24).

Case (*iii*): $x_2 \neq x_1$; $x_2 \neq x_3$. The useful coordinates are $X_{31} = x_1 - x_3$, $Y_{31} = x_1 + x_3$ and $Z_{31} = x_2$. Again the equation can be reduced to the form in (25) and solved explicitly.

It is surprising that in case of DNLSE, a very simple pattern has emerged regarding the wavefunction structure of the Bethe states. From the well known rule of [3] one can perhaps conclude that the n-particle states will also be decomposable as in the case of two- and three-particle states. So even though, due to the non-canonical and non-ultralocal character, it was not possible to quantise this non-linear system via QISM, the Bethe states may yet be easily constructed and full information regarding the quantised non-linear system can be obtained from them.

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